GRAVITATIONAL INTERACTION OF HIGHER SPIN MASSIVE FIELDS AND STRING THEORY

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We discuss the problem of consistent description of higher spin massive fields coupled to external gravity. As an example we consider massive field of spin 2 in arbitrary gravitational field. Consistency requires the theory to have the same number of degrees of freedom as in flat spacetime and to describe causal propagation. By careful analysis of lagrangian structure of the theory and its constraints we show that there exist at least two possibilities of achieving consistency. The first possibility is provided by a lagrangian on specific manifolds such as static or Einstein spacetimes. The second possibility is realized in arbitrary curved spacetime by a lagrangian representing an infinite series in curvature. In the framework of string theory we derive equations of motion for background massive spin 2 field coupled to gravity from the requirement of quantum Weyl invariance. These equations appear to be a particular case of the general consistent equations obtained from the field theory point of view.

1 Introduction

Despite many years of intensive studies the construction of consistent interacting theories of higher spin fields is still far from completion. Consistency problems arise both in higher spin field theories with self-interaction and in models of a single higher spin field in non-trivial external background. In this contribution we review a recent progress[1,2] achieved in building of a consistent theory of massive spin 2 field in external gravity and in understanding about the way the string theory predicts consistent equations of motion for such a system.

In general, there are two ways the interaction can spoil the consistency of a higher spin fields theory. Firstly, interaction may change the number of dynamical degrees of freedom. For example, a massive field with spin s

in $D=4$ Minkowski spacetime is described by a rank s symmetric traceless transverse tensor $\phi_{(\mu_1\ldots\mu_s)}$ satisfying the mass shell condition:

$$(\partial^2-m^2)\phi_{\mu_1\ldots\mu_s}=0, \qquad \partial^\mu\phi_{\mu\mu_1\ldots\mu_{s-1}}=0, \qquad \phi^\mu{}_{\mu\mu_1\ldots\mu_{s-2}}=0. \quad (1)$$

To reproduce all these equations from a single lagrangian one needs to introduce auxiliary fields $\chi_{\mu_1\ldots\mu_{s-2}}$, $\chi_{\mu_1\ldots\mu_{s-3}}$, $\ldots$, χ[3,4]. These symmetric traceless fields vanish on shell but their presence in the theory provides lagrangian description of the conditions (1). In higher dimensional spacetimes there appear fields of more complex tensor structure but general situation remains the same, i.e. lagrangian description always requires presence of unphysical auxiliary degrees of freedom.

Namely these auxiliary fields create problems when one tries to turn on interaction in the theory. Arbitrary interaction makes the auxiliary fields dynamical thus increasing the number of degrees of freedom. Usually these extra degrees of freedom are ghostlike and should be considered as pathological. Requirement of absence of these extra dynamical degrees of freedom imposes severe restrictions on the possible interaction[5,6,7,8,9].

The other problem that may arise in higher spin fields theories is connected with possible violation of causal properties. This problem was first noted in the theory of spin 3/2 field in external fields[10] (see also the review[11] and a recent discussion in[12])

In general, when one has a system of differential equations for a set of fields ϕ^B (to be specific, let us say about second order equations)

$$M_{AB}{}^{\mu\nu}\partial_\mu\partial_\nu\phi^B+\ldots=0, \qquad \mu,\nu=0,\ldots,D-1 \qquad (2)$$

the following definitions are used. A characteristic matrix is the matrix function of D arguments n_μ built out of the coefficients at the second derivatives in the equations: $M_{AB}(n)=M_{AB}{}^{\mu\nu}n_\mu n_\nu$. A characteristic equation is $\det M_{AB}(n)=0$. A characteristic surface is the surface $S(x)=const$ where $\partial_\mu S(x)=n_\mu$.

If for any n_i $(i=1,\ldots,D-1)$ all solutions of the characteristic equation $n_0(n_i)$ are real then the system of differential equations is called hyperbolic and describes propagation of some wave processes. The hyperbolic system is called causal if there is no timelike vectors among solutions n_μ of the characteristic equations. Such a system describes propagation with a velocity not exceeding the speed of light. If there exist timelike solutions for n_μ then the corresponding characteristic surfaces are spacelike and violate causality.

Turning on interaction in theories of higher spin fields in general changes the characteristic matrix and there appears possibility of superluminal propagation. Such a situation also should be considered as pathological. Note that the requirement of causal behaviour is an independent condition. Interaction with external fields may violate causality even in a covariant theory with the correct number of degrees of freedom.

As an example where both these problem arise we consider the theory of massive spin 2 field in external gravitational field in arbitrary spacetime dimension. In Section 2 we describe the structure of lagrangian equations of motion and constraints in such a theory and demonstrate how correct number of degrees of freedom can be achieved in a number of specific spacetimes. Namely, we consider two examples - an arbitrary static spacetime and an Einstein spacetime. In both cases there exists correct flat spacetime limit though in static case there may be regions where propagation of some of the spin 2 field comonents is acausal.

Another possibility of achieving consistency is described in Section 3 where lagrangian equations of motion for massive spin 2 field are constructed in form of infinite series in curvature. These kinds of infinite series arise naturally in string theory which contains an infinite tower of massive higher spin excitations and so should also provide a consistent scheme for description of higher spin fields interaction.

Section 4 is devoted to open string theory in background of massless graviton and massive spin 2 field. As is well known[13] the requirement of quantum Weyl invariance of two-dimensional $\sigma-$model coupled to massless background fileds gives rise to effective equations of motion for these fields. In case of massive background fields the coresponding $\sigma-$model action is non-renormalizable and should contain an infinite number of terms but as was shown in[14] a specific structure of renormalization makes it possible to calculate all $\beta-$functions pertaining in each perturbative order only finite number of counterterms. In linear order the effective equations of motion obtained that way were shown to be in agreement with canonical analysis of the corresponding $\sigma-$model action[15]. In this contribution we show that string theory also gives consistent equations of motion for massive spin 2 field interacting with gravity which represent a particular case of general equations described in the previous sections.

2 Massive spin 2 field on specific manifolds

Let us start with reminding the lagrangian structure of a free massive spin 2 field. To find the complete set of constraints we use the general lagrangian scheme[16] which is equivalent to the Dirac-Bergmann procedure in hamiltonian formalism but for our purposes is simpler. In the case of second class constraints (which is relevant for massive higher spin fields) it consists in the following steps. If in a theory of some set of fields $\phi^A(x)$, $A = 1, \ldots, N$ the original lagrangian equations of motion define only $r < N$ of the second time derivatives ("accelerations") $\ddot{\phi}^A$ then one can build $N - r$ primary constraints, i.e. linear combinations of the equations of motion that do not contain accelerations. Requirements of conservation in time of the primary constraints either define some of the missing accelerations or lead to new (secondary) constraints. Then one demands conservation of the secondary constraints and so on, until all the accelerations are defined and the procedure closes up.

In the flat spacetime the massive spin 2 field is described (as follows from the analysis of irreducible representations of 4-dimensional Poincare group) by symmetric transversal and traceless tensor of the second rank $H_{\mu\nu}$ satisfying mass-shell condition:

$$\left(\partial^2 - m^2\right) H_{\mu\nu} = 0, \qquad \partial^\mu H_{\mu\nu} = 0, \qquad H^\mu{}_\mu = 0. \tag{3}$$

In higher dimensional spacetimes Poincare algebras have more than two Casimir operators and so there are several different spins for $D > 4$. Talking about spin 2 massive field in arbitrary dimension we will mean, as usual, that this field by definition satisfies the same equations (3) as in $D = 4$. After dimensional reduction to $D = 4$ such a field will describe massive spin two representation of $D = 4$ Poincare algebra plus infinite tower of Kaluza-Klein descendants.

All the equations (3) can be derived from the Fierz-Pauli action[3]:

$$\begin{aligned} S = \int d^D x \Big\{ & \frac{1}{4}\partial_\mu H \partial^\mu H - \frac{1}{4}\partial_\mu H_{\nu\rho}\partial^\mu H^{\nu\rho} - \frac{1}{2}\partial^\mu H_{\mu\nu}\partial^\nu H + \frac{1}{2}\partial_\mu H_{\nu\rho}\partial^\rho H^{\nu\mu} \\ & - \frac{m^2}{4} H_{\mu\nu} H^{\mu\nu} + \frac{m^2}{4} H^2 \Big\} \end{aligned} \tag{4}$$

where $H = \eta^{\mu\nu} H_{\mu\nu}$.

Here the role of auxiliary field is played by the trace $H = \eta^{\mu\nu} H_{\mu\nu}$. The equations of motion

$$\begin{aligned} E_{\mu\nu} = {} & \partial^2 H_{\mu\nu} - \eta_{\mu\nu}\partial^2 H + \partial_\mu\partial_\nu H + \eta_{\mu\nu}\partial^\alpha\partial^\beta H_{\alpha\beta} - \partial_\sigma\partial_\mu H^\sigma{}_\nu - \partial_\sigma\partial_\nu H^\sigma{}_\mu \\ & - m^2 H_{\mu\nu} + m^2 H \eta_{\mu\nu} = 0 \end{aligned} \tag{5}$$

contain D primary constraints (expressions without second time derivatives $\ddot{H}_{\mu\nu}$):

$$E_{00} = \Delta H_{ii} - \partial_i\partial_j H_{ij} - m^2 H_{ii} \equiv \varphi_0^{(1)} \approx 0 \tag{6}$$

$$E_{0i} = \Delta H_{0i} + \partial_i \dot{H}_{kk} - \partial_k \dot{H}_{ki} - \partial_i\partial_k H_{0k} - m^2 H_{0i} \equiv \varphi_i^{(1)} \approx 0. \tag{7}$$

The remaining equations of motion $E_{ij} = 0$ allow to define the accelerations $\ddot{H}_{ij}$ in terms of $\dot{H}_{\mu\nu}$ and $H_{\mu\nu}$. The accelerations $\ddot{H}_{00}$, $\ddot{H}_{0i}$ cannot be expressed from the equations directly.

Conditions of conservation of the primary constraints in time $\dot{E}_{0\mu} \approx 0$ lead to D secondary constraints. On-shell they are equivalent to

$$\varphi^{(2)}_\nu = \partial^\mu E_{\mu\nu} = m^2 \partial_\nu H - m^2 \partial^\mu H_{\mu\nu} \approx 0 \tag{8}$$

Conservation of $\varphi^{(2)}_i$ defines $D-1$ accelerations $\ddot{H}_{0i}$ and conservation of $\varphi^{(2)}_0$ gives another one constraint. It is convenient to choose it in the covariant form by adding suitable terms proportional to the equations of motion:

$$\varphi^{(3)} = \partial^\mu \partial^\nu E_{\mu\nu} + \frac{m^2}{D-2} \eta^{\mu\nu} E_{\mu\nu} = H m^4 \frac{D-1}{D-2} \approx 0 \tag{9}$$

Conservation of $\varphi^{(3)}$ gives one more constraint on initial values

$$\varphi^{(4)} = -\dot{H}_{00} + \dot{H}_{kk} = \dot{H} \approx 0 \tag{10}$$

and from the conservation of this last constraint the acceleration $\ddot{H}_{00}$ is defined. Altogether there are $2D+2$ constraints on the initial values of $\dot{H}_{\mu\nu}$ and $H_{\mu\nu}$.

Obviously, the equations of motion (3) are causal because the characteristic equation

$$\det M(n) = (n^2)^{D(D+1)/2} \tag{11}$$

has 2 multiply degenerate roots

$$-n_0^2 + n_i^2 = 0, \qquad n_0 = \pm\sqrt{n_i^2}. \tag{12}$$

which correspond to real null solutions for n_μ. Note that analysis of causality is possible only after calculation of all the constraints. Original lagrangian equations of motion (5) have degenerate characteristic matrix $\det M(n) \equiv 0$ and do not allow to define propagation cones of the field $H_{\mu\nu}$.

Now if we want to construct a theory of massive spin 2 field on a curved manifold we should provide the same number of propagating degrees of freedom as in the flat case. It means that new equations of motion $E_{\mu\nu}$ should lead to exactly $2D+2$ constraints and in the flat spacetime limit these constraints should reduce to their flat counterparts. The important point here is that consistency does not require any specific transformation properties of constraints in curved spacetime. For example, in flat case the constraints $\varphi^{(2)}_\mu$ form a Lorentz vector but there is no reason to require their curved counterpart to be a vector with respect to local Lorentz transformation. The only conditions one should care of is that the total number of constraints should conserve and that they should always be of the second class. In massless higher spin fields theory one should also require conservation of the corresponding gauge algebra and achieving consistency in that case is a more difficult task[17].

Generalizing (4) to curved spacetime we should substitute all derivatives by the covariant ones and also we can add non-minimal terms containing curvature tensor with some dimensionless coefficients in front of them. As a result, the most general action for massive spin 2 field in curved spacetime quadratic in derivatives and consistent with the flat limit should have the form[5]:

$$
\begin{aligned}
S = \int d^D x\sqrt{-G}\Big\{ & \frac{1}{4}\nabla_\mu H \nabla^\mu H - \frac{1}{4}\nabla_\mu H_{\nu\rho}\nabla^\mu H^{\nu\rho} - \frac{1}{2}\nabla^\mu H_{\mu\nu}\nabla^\nu H \\
& + \frac{1}{2}\nabla_\mu H_{\nu\rho}\nabla^\rho H^{\nu\mu} + \frac{a_1}{2}R H_{\alpha\beta}H^{\alpha\beta} + \frac{a_2}{2}R H^2 + \frac{a_3}{2}R^{\mu\alpha\nu\beta}H_{\mu\nu}H_{\alpha\beta} \\
& + \frac{a_4}{2}R^{\alpha\beta}H_{\alpha\sigma}H_\beta{}^\sigma + \frac{a_5}{2}R^{\alpha\beta}H_{\alpha\beta}H - \frac{m^2}{4}H_{\mu\nu}H^{\mu\nu} + \frac{m^2}{4}H^2\Big\}
\end{aligned}
\tag{13}
$$

where $a_1, \dots a_5$ are so far arbitrary dimensionless coefficients, $R^\mu{}_{\nu\lambda\kappa} = \partial_\lambda \Gamma^\mu_{\nu\kappa} - \dots$, $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$.

Equations of motion contain second time derivatives of $H_{\mu\nu}$ in the following way:

$$
\begin{aligned}
E_{00} &= (G^{mn} - G_{00}G^{00}G^{mn} + G_{00}G^{0m}G^{0n})\nabla_0\nabla_0 H_{mn} + O(\nabla_0), \\
E_{0i} &= (-G_{0i}G^{00}G^{mn} + G_{0i}G^{0m}G^{0n} - G^{0m}\delta_i^n)\nabla_0\nabla_0 H_{mn} + O(\nabla_0), \\
E_{ij} &= (G^{00}\delta_i^m\delta_j^n - G_{ij}G^{00}G^{mn} + G_{ij}G^{0m}G^{0n})\nabla_0\nabla_0 H_{mn} + O(\nabla_0)
\end{aligned}
\tag{14}
$$

So we see that accelerations $\ddot{H}_{00}$ and $\ddot{H}_{0i}$ again (as in the flat case) do not enter the equations of motion while accelerations $\ddot{H}_{ij}$ can be expressed through $\dot{H}_{\mu\nu}$, $H_{\mu\nu}$ and their spatial derivatives.

There are D linear combinations of the equations of motion which do not contain second time derivatives and so represent primary constraints of the theory:

$$
\varphi^{(1)}_\mu = E^0{}_\mu = G^{00}E_{0\mu} + G^{0j}E_{j\mu} \tag{15}
$$

At the next step one should calculate time derivatives of these constraints and define secondary ones. In order to do this in a covariant form we add to the time derivative of $\varphi^{(1)}_\mu$ a linear combination of equations of motion and primary constraints and define the secondary constraints as follows:

$$
\varphi^{(2)}_\mu = \nabla^\alpha E_{\alpha\mu} \tag{16}
$$

Conservation of these D secondary constraints should lead to one new constraint and to expressions for $D-1$ accelerations $\ddot{H}_{0i}$. This means that the constraints (16) should contain the first time derivatives $\dot{H}_{0\mu}$ through the matrix $\hat{\Phi}_\mu{}^\nu$ built out of the blocks A, B^j, C_i, $D_i{}^j$

$$\begin{aligned}\varphi_0^{(2)} &= A\,\dot{H}_{00} + B^j\dot{H}_{0j} + \ldots \\ \varphi_i^{(2)} &= C_i\dot{H}_{00} + D_i{}^j\dot{H}_{0j} + \ldots\end{aligned} \tag{17}$$

whose rank is equal to $D-1$.

In the flat spacetime we had the matrix block elements

$$A = B^j = C_i = 0, \qquad D_i{}^j = m^2\delta_i^j \tag{18}$$

while in the curved case the explicit form of these elements in the constraints (16) is:

$$\begin{aligned}A &= RG^{00}(2a_1+2a_2) + R^{00}(a_4+a_5) + R^0{}_0G^{00}(a_4+a_5-1) \\ B^j &= m^2G^{0j} + RG^{0j}(2a_1+4a_2) + 2a_3R^{0j}{}_0{}^0 + R^j{}_0G^{00}(a_4-2) \\ &\quad + R^{0j}(a_4+2a_5) + R^0{}_0G^{0j}(a_4+2a_5) \\ C_i &= R^0{}_iG^{00}(a_4+a_5-1) \\ D_i{}^j &= -m^2G^{00}\delta_i^j + 2a_1RG^{00}\delta_i^j + 2a_3R^{0j}{}_i{}^0 + a_4R^{00}\delta_i^j \\ &\quad + (a_4-2)R^j{}_iG^{00} + (a_4+2a_5)R_i^0G^{0j}\end{aligned} \tag{19}$$

At this stage the restrictions that consistency imposes on the type of interaction reduce to the requirements that the above matrix elements give

$$\det\hat{\Phi} = 0, \qquad \det D_i{}^j \neq 0 \tag{20}$$

When the gravitational background is arbitrary it is not clear how to fulfill this condition by choosing some specific values of non-minimal couplings a_1, $\ldots a_5$. For example, requirement of vanishing of the elements A and C_i (19) would lead to contradictory equations $a_4+a_5=0$, $a_4+a_5-1=0$.

But the consistency conditions (20) can be fulfilled in a number of specific gravitational background. Namely, any spacetime which in some coordinates has

$$R^0{}_i = 0 \tag{21}$$

provides such an example. In such a spacetime $R^{00} = R^0{}_0G^{00}$ and choosing coefficients $a_1+a_2=0$, $2a_4+2a_5=1$ we have the first column of the matrix $\hat{\Phi}$ vanishing and so the conditions (20) fulfilled.

As a first example where (21) holds let us consider an arbitrary static spacetime, i.e. a spacetime having a timelike Killing vector and invariant with respect to the time reversal $x^0 \to -x^0$. In such a spacetime one can always find coordinates where

$$\partial_0 G_{\mu\nu} = 0, \qquad G_{0i} = 0. \tag{22}$$

The matrix elements (19) in this case become

$$\begin{aligned} &A = RG^{00}(2a_1 + 2a_2) + R^{00}(2a_4 + 2a_5 - 1), \\ &B^j = 0, \qquad\qquad C_i = 0, \\ &D_i{}^j = (-m^2 G^{00} + 2a_1 RG^{00} + a_4 R^{00})\delta_i^j + (a_4 - 2)R^j{}_i G^{00} + 2a_3 R^{0j}{}_i{}^0 \end{aligned} \tag{23}$$

and (20) lead to the following conditions:

$$2a_1 + 2a_2 = 0, \qquad 2a_4 + 2a_5 - 1 = 0, \qquad \det D_i{}^j \neq 0 \tag{24}$$

The last inequality may be violated in strong gravitational field and as we comment below this fact may lead to causal problems.

Suppose that all the conditions (24) are fulfilled. For simplicity we also choose $a_3 = 0$. Then we have the classical action of the form (13) with the coefficients

$$a_1 = \frac{\xi_1}{2}, \quad a_2 = -\frac{\xi_1}{2}, \quad a_3 = 0, \quad a_4 = \frac{1}{2} - \xi_2, \quad a_5 = \xi_2 \tag{25}$$

where ξ_1, ξ_2 are two arbitrary coupling parameters.

One of the secondary constraints

$$\varphi_0^{(2)} = \nabla^\alpha E_{\alpha 0} \tag{26}$$

does not contain velocities $\dot{H}_{00}$, $\dot{H}_{0i}$ and so its conservation leads to a new constraint $\varphi^{(3)} \approx \nabla_0 \nabla^\alpha E_{\alpha 0}$. After exclusion from this expression the accelerations $\ddot{H}_{ij}$ we get this constraint as the following combination of the equations of motion:

$$\begin{aligned} \varphi^{(3)} &= \nabla_0 \nabla^\mu E_{\mu 0} - \xi_2 G_{00} R^{ij} E_{ij} \\ &\quad + \frac{1}{D-2}\Big[m^2 G_{00} + (\xi_2 - \xi_1) R G_{00} + R_{00}\Big] G^{ij} E_{ij} \end{aligned} \tag{27}$$

$\varphi^{(3)}$ contains neither the acceleration $\ddot{H}_{00}$ nor the velocity $\dot{H}_{00}$. It means that its conservation in time leads to another new constraints

$$\varphi^{(4)} \approx \nabla_0 \varphi^{(3)} \tag{28}$$

and hence the total number of constraints is the same as in the flat spacetime.

Unfortunately, analysis of causal properties of such a theory on static background[2] shows that there can be spacetime regions where some of the above constraints fail to be of the second class and some components of $H_{\mu\nu}$ may propagate with superluminal velocities.

Another possible way to fulfill the consistency requirements (20) is to consider spacetimes representing solutions of vacuum Einstein equations with arbitrary cosmological constant:

$$R_{\mu\nu} = \frac{1}{D} G_{\mu\nu} R \,. \tag{29}$$

In this case the coefficients a_4, a_5 in the lagrangian (13) are absent and the elements of the matrix $\hat{\Phi}$ take the form:

$$\begin{aligned}
A &= RG^{00}(2a_1 + 2a_2 - \frac{1}{D}) \\
B^j &= RG^{0j}(2a_1 + 4a_2) + 2a_3 R^{0j}{}_0{}^0 + m^2 G^{0j} \\
C_i &= 0 \\
D_i{}^j &= 2a_3 R^{0j}{}_i{}^0 + G^{00}\delta_i^j (2a_1 - \frac{2}{D}) - m^2 G^{00} \delta_i^j
\end{aligned} \tag{30}$$

The simplest way to make the rank of such a matrix to be equal to $D-1$ is provided by the following choice of the coefficients:

$$2a_1 + 2a_2 - \frac{1}{D} = 0, \qquad a_3 = 0, \qquad 2R\left(a_1 - \frac{1}{D}\right) - m^2 \neq 0. \tag{31}$$

As a result, we have one-parameter family of theories:

$$\begin{aligned}
&a_1 = \frac{\xi}{D}, \quad a_2 = \frac{1-2\xi}{2D}, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0 \\
&R_{\mu\nu} = \frac{1}{D} G_{\mu\nu} R, \qquad \frac{2(1-\xi)}{D} R + m^2 \neq 0.
\end{aligned} \tag{32}$$

with ξ an arbitrary real number.

The action in this case takes the form

$$\begin{aligned}
S = \int d^D x \sqrt{-G} \Big\{ &\frac{1}{4} \nabla_\mu H \nabla^\mu H - \frac{1}{4} \nabla_\mu H_{\nu\rho} \nabla^\mu H^{\nu\rho} - \frac{1}{2} \nabla^\mu H_{\mu\nu} \nabla^\nu H \\
&+ \frac{1}{2} \nabla_\mu H_{\nu\rho} \nabla^\rho H^{\nu\mu} + \frac{\xi}{2D} R H_{\mu\nu} H^{\mu\nu} + \frac{1-2\xi}{4D} R H^2 \\
&- \frac{m^2}{4} H_{\mu\nu} H^{\mu\nu} + \frac{m^2}{4} H^2 \Big\}.
\end{aligned} \tag{33}$$

and the corresponding equations of motion are

$$E_{\mu\nu} = \nabla^2 H_{\mu\nu} - G_{\mu\nu} \nabla^2 H + \nabla_\mu \nabla_\nu H + G_{\mu\nu} \nabla^\alpha \nabla^\beta H_{\alpha\beta} - \nabla_\sigma \nabla_\mu H^\sigma{}_\nu$$

$$- \nabla_\sigma \nabla_\nu H^\sigma{}_\mu + \frac{2\xi}{D} R H_{\mu\nu} + \frac{1-2\xi}{D} R H G_{\mu\nu}$$
$$- m^2 H_{\mu\nu} + m^2 H G_{\mu\nu} = 0 \tag{34}$$

The secondary constraints built out of them are

$$\varphi^{(2)}_\mu = \nabla^\alpha E_{\alpha\mu} = (\nabla_\mu H - \nabla^\alpha H_{\mu\alpha}) \left(m^2 + \frac{2(1-\xi)}{D} R \right) \tag{35}$$

Just like in the flat case, in this theory the conditions $\dot\varphi^{(2)}_i \approx 0$ define the accelerations $\ddot H_{0i}$ and the condition $\dot\varphi^{(2)}_0 \approx 0$ after excluding $\ddot H_{0i}$ gives a new constraint, i.e. the acceleration $\ddot H_{00}$ is not defined at this stage.

To define the new constraint in a covariant form we use the following linear combination of $\dot\varphi^{(2)}_\mu$, equations of motion, primary and secondary constraints:

$$\varphi^{(3)} = \frac{m^2}{D-2} G^{\mu\nu} E_{\mu\nu} + \nabla^\mu \nabla^\nu E_{\mu\nu} + \frac{2(1-\xi)}{D(D-2)} R G^{\mu\nu} E_{\mu\nu} = \tag{36}$$
$$= H \frac{1}{D-2} \left(\frac{2(1-\xi)}{D} R + m^2 \right) \left(\frac{D + 2\xi(1-D)}{D} R + m^2 (D-1) \right)$$

This gives tracelessness condition for the field $H_{\mu\nu}$ provided that parameters of the theory fulfill the conditions:

$$\frac{2(1-\xi)}{D} R + m^2 \neq 0, \qquad \frac{D + 2\xi(1-D)}{D} R + m^2 (D-1) \neq 0 \tag{37}$$

Requirement of conservation of $\varphi^{(3)}$ leads to one more constraint

$$\dot\varphi^{(3)} \sim \dot H \quad \Longrightarrow \quad \varphi^{(4)} = \dot H \approx 0. \tag{38}$$

The last acceleration $\ddot H_{00}$ is expressed from the condition $\dot\varphi^{(4)} \approx 0$.

Using the constraints for simplifying the equations of motion we see that the original equations are equivalent to the following system:

$$\begin{aligned}
&\nabla^2 H_{\mu\nu} + 2 R^\alpha{}_\mu{}^\beta{}_\nu H_{\alpha\beta} + \frac{2(\xi-1)}{D} R H_{\mu\nu} - m^2 H_{\mu\nu} = 0, \\
&H^\mu{}_\mu = 0, \qquad \dot H^\mu{}_\mu = 0, \qquad \nabla^\mu H_{\mu\nu} = 0, \\
&G^{00} \nabla_0 \nabla_i H^i{}_\nu - G^{0i} \nabla_0 \nabla_i H^0{}_\nu - G^{0i} \nabla_i \nabla_0 H^0{}_\nu - G^{ij} \nabla_i \nabla_j H^0_\nu \\
&\qquad - 2 R^{\alpha 0 \beta}{}_\nu H_{\alpha\beta} - \frac{2(\xi-1)}{D} R H^0{}_\nu + m^2 H^0{}_\nu = 0.
\end{aligned} \tag{39}$$

The last expression represents D primary constraints.

For any values of ξ (except two degenerate values excluded by (37)) the theory describes the same number of degrees of freedom as in the flat case - the symmetric, covariantly transverse and traceless tensor. D primary constraints guarantees conservation of the transversality conditions in time.

Let us now consider the causal properties of the theory. Again, if we tried to use the equations of motion in the original lagrangian form (34) then the characteristic matrix would be degenerate. After having used the constraints we obtain the equations of motion written in the form (39) and the characteristic matrix becomes non-degenerate:

$$M_{\mu\nu}{}^{\lambda\kappa}(n) = \delta_{\mu\nu}{}^{\lambda\kappa} n^2, \qquad n^2 = G^{\alpha\beta} n_\alpha n_\beta. \tag{40}$$

The characteristic cones remains the same as in the flat case. At any point x_0 we can choose locally $G^{\alpha\beta}(x_0) = \eta^{\alpha\beta}$ and then

$$n^2\big|_{x_0} = -n_0^2 + n_i^2 \tag{41}$$

Just like in the flat case the equations are hyperbolic and causal.

Now let us discuss the massless limit of the theory under consideration. There are several points of view on the definition of masslessness in a curved spacetime of an arbitrary dimension. We guess that the most physically accepted definition is the one referring to appearance of a gauge invariance for some specific values of the theory parameters (see e.g.[19,20] for a recent discussion).

In our case it means that the real mass parameter M for the field $H_{\mu\nu}$ in an Einstein spacetime is defined as

$$M^2 = m^2 + \frac{2(1-\xi)}{D} R \tag{42}$$

When $M^2 = 0$ instead of D secondary constraints $\varphi_\mu^{(2)}$ we have D identities for the equations of motion $\nabla^\mu E_{\mu\nu} \equiv 0$ and the theory acquires gauge invariance $\delta H_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. This explains the meaning of the first condition in (37), it just tells us that the theory is massive.

In fact, two parameters m^2 and ξ enter the action (33) in a single combination M^2 (42). Since scalar curvature is constant in Einstein spacetime there is no way to distinguish between the corresponding terms $\sim \xi RHH$, $\sim m^2 HH$ (with arbitrary ξ, m) in the action. The difference between the two will appear only if we consider Weyl rescaling of the metric. Note that the "massless" theory with $M^2 = 0$ is not Weyl invariant. In the case of dS/AdS spacetimes the difference between masslessness, conformal and gauge invariance and null cone propagation was discussed in detail in[21]. In our case the theory obviously cannot possess Weyl invariance.

The second inequality (37) is more mysterious. If it fails to hold, i.e. if $M^2 = M_c^2 \equiv \frac{D-2}{D(D-1)}R$ then instead of the constraint $\varphi^{(3)}$ the scalar identity

$$\nabla^\mu \nabla^\nu E_{\mu\nu} + \frac{R}{D(D-1)} G^{\mu\nu} E_{\mu\nu} = 0 \tag{43}$$

with the corresponding gauge invariance

$$\delta H_{\mu\nu} = \nabla_\mu \nabla_\nu \epsilon + \frac{R}{D(D-1)} G_{\mu\nu} \epsilon \tag{44}$$

arise.

Appearance of this gauge invariance with a scalar parameter was first found for the massive spin 2 in spacetime of constant curvature in[21] and was further investigated[6,7] in spacetimes with positive cosmological constant. Our analysis shows that this gauge invariance is a feature of more general spin 2 theories in arbitrary Einstein spacetimes. In this case we can simplify the equations of motion using the secondary constraints (35):

$$\nabla^2 H_{\mu\nu} - \nabla_\mu \nabla_\nu H + 2R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + \frac{2-D}{D(D-1)} R H_{\mu\nu} - \frac{1}{D(D-1)} R G_{\mu\nu} H = 0. \tag{45}$$

After imposing the gauge condition[a] $H = 0$ one can see that these equations describe causal propagation of the field $H_{\mu\nu}$ but the number of propagating degrees of freedom corresponds to neither massive nor massless spin 2 free field. It was argued in[6,7] that appearance of the gauge invariance (44) leads to such pathological properties as violation of the classical Hamiltonian positiveness and negative norm states in the quantum version of the theory. One should expect similar problems in the general spin 2 theory in arbitrary Einstein spacetime described in this paper.

3 Consistent equations in arbitrary gravitational background

In the previous section we analyzed a possibility of consistent description of the spin 2 field on specific spacetime manifolds. Now we will describe another possibility which allows to remove any restrictions on the external gravitational background by means of considering a lagrangian in the form of infinite series in inverse mass m. Existence of dimensionful mass parameter m in the theory let us construct a lagrangian with terms of arbitrary orders

[a]It does not fix (44) completely and the residual symmetry with the prameter obeying $\left(\nabla^2 + \frac{R}{D-1}\right)\epsilon = 0$ remains.

in curvature multiplied by the corresponding powers of $1/m^2$, i.e having the following schematic form:

$$S_H = \int d^D x \sqrt{-G} \Big\{ \nabla H \nabla H + RHH + m^2 HH + \frac{1}{m^2}(R\nabla H \nabla H + RH\nabla\nabla H + RRHH) + O\Big(\frac{1}{m^4}\Big) \Big\} \qquad (46)$$

Actions of this kind are expected to arise naturally in string theory where the role of mass parameter is played by string tension $m^2 = 1/\alpha'$ and perturbation theory in α' will give for background fields effective actions of the form (46). Possibility of constructing consistent equations for massive higher spin fields as series in curvature was recently studied in[9] where such equations were derived in particular case of symmetrical Einstein spaces in linear in curvature order.

Here we demonstrate that requirement of consistency with the flat spacetime limit can be fulfilled perturbatively in $1/m^2$ for arbitrary gravitational background at least in the lowest order. We use the same general scheme of calculating lagrangian constraints as in the previous section. The only difference is that each condition will be considered perturbatively and can be solved separately in each order in $1/m^2$.

Primary constraints in the theory described by the action (46) should be given by the equations ${E^0}_\mu \approx 0$. Requirement of absence of second time derivatives in these equations will give some restrictions on coefficients in higher orders in $1/m^2$, for example, in terms like $R\nabla H \nabla H$.

Consistency with the flat spacetime limit requires existence of one additional constraint among conservation conditions of the secondary constraints. The advantage of having a theory in the form of infinite series consists in the possibility to calculate the determinant of the matrix $\hat{\Phi}$ perturbatively in $1/m^2$. Assuming that the lower right subdeterminant of the matrix is not zero (it is not zero in the flat case) one has

$$\det \hat{\Phi} = (A - BD^{-1}C) \det D\,, \qquad \det D \neq 0 \qquad (47)$$

Converting the matrix D perturbatively

$$D^{-1} = -\frac{1}{m^2 G^{00}} \delta_i^j + O\left(\frac{1}{m^4}\right) \qquad (48)$$

we get

$$A - BD^{-1}C = RG^{00} 2(a_1 + a_2) + R^{00}(2a_4 + 2a_5 - 1) + O\left(\frac{1}{m^2}\right) \qquad (49)$$

So consistency with the flat limit imposes at this order in m^2 two conditions on the five non-minimal couplings in the lagrangian (46) and we are left with a three parameters family of theories:

$$a_1 = \frac{\xi_1}{2}, \quad a_2 = -\frac{\xi_1}{2}, \quad a_3 = \frac{\xi_3}{2}, \quad a_4 = \frac{1}{2} - \xi_2, \quad a_5 = \xi_2. \tag{50}$$

The action (46) then takes the form:

$$\begin{aligned} S_H = \int d^D x \sqrt{-G} \Big\{ & \frac{1}{4}\nabla_\mu H \nabla^\mu H - \frac{1}{4}\nabla_\mu H_{\nu\rho}\nabla^\mu H^{\nu\rho} - \frac{1}{2}\nabla^\mu H_{\mu\nu}\nabla^\nu H \\ & + \frac{1}{2}\nabla_\mu H_{\nu\rho}\nabla^\rho H^{\nu\mu} + \frac{\xi_1}{4} R H_{\alpha\beta}H^{\alpha\beta} - \frac{\xi_1}{4} R H^2 + \frac{1-2\xi_2}{4} R^{\alpha\beta} H_{\alpha\sigma} H_\beta{}^\sigma \\ & + \frac{\xi_2}{2} R^{\alpha\beta} H_{\alpha\beta} H + \frac{\xi_3}{2} R^{\mu\alpha\nu\beta} H_{\mu\nu} H_{\alpha\beta} - \frac{m^2}{4} H_{\mu\nu}H^{\mu\nu} + \frac{m^2}{4} H^2 \\ & + O\Big(\frac{1}{m^2}\Big) \Big\} \end{aligned} \tag{51}$$

In this case the rank of the matrix $\hat{\Phi}$ is equal to $D-1$ and one can construct from the conservation conditions for the secondary constraints

$$\nabla_0 \varphi^{(2)}_\nu = \nabla_0 \nabla^\mu E_{\mu\nu} = \hat{\Phi}_\nu{}^\mu \ddot{H}_{0\mu} + \ldots \tag{52}$$

one covariant linear combination which does not contain acceleration $\ddot{H}_{0\mu}$:

$$\varphi^{(3)} \approx \nabla^\mu \nabla^\nu E_{\mu\nu} - \frac{1}{m^2} R^{\alpha\nu} \nabla_\alpha \nabla^\mu E_{\mu\nu} + O\Big(\frac{1}{m^4}\Big) \tag{53}$$

Derivatives of the field $H_{\mu\nu}$ enter this expression in such a way that it does not contain the accelerations $\ddot{H}_{00}$, $\ddot{H}_{0\mu}$ and the velocity $\dot{H}_{00}$. It means that just like in the flat case the conservation condition $\dot{\varphi}^{(3)} \approx 0$ leads to another new constraints $\varphi^{(4)}$ and the last acceleration $\ddot{H}_{00}$ is defined from $\dot{\varphi}^{(4)} \approx 0$. The total number of constraints coincides with that in the flat spacetime.

The constraints $\varphi^{(3)}$ and $\varphi^{(2)}_\mu$ can be solved perturbatively in $1/m^2$ with respect to the trace and the longitudinal part of $H_{\mu\nu}$

$$\varphi^{(3)} \sim H + O\Big(\frac{1}{m^2}\Big), \qquad \varphi^{(2)}_\mu \sim \nabla^\mu H_{\mu\nu} + O\Big(\frac{1}{m^2}\Big) \tag{54}$$

and used for reducing the original equations of motion to the conditions:

$$\begin{aligned} & \nabla^2 H_{\mu\nu} - m^2 H_{\mu\nu} + \xi_1 R H_{\mu\nu} - (\frac{1}{2} + \xi_2)(R_\mu{}^\alpha H_{\alpha\nu} + R_\nu{}^\alpha H_{\alpha\mu}) \\ & + (\xi_3 + 2) R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + \frac{\xi_2 - \xi_3 - 1}{D-1} G_{\mu\nu} R^{\alpha\beta} H_{\alpha\beta} + O\Big(\frac{1}{m^2}\Big) = 0, \\ & H + O\Big(\frac{1}{m^2}\Big) = 0, \qquad \nabla^\mu H_{\mu\nu} + O\Big(\frac{1}{m^2}\Big) = 0 \end{aligned} \tag{55}$$

and also to the D primary constraints $E^0{}_\mu$. We see that even in this lowest order in m^2 not all non-minimal terms in the equations are arbitrary. Consistency with the flat limit leaves only three arbitrary parameters while the number of different non-minimal terms in the equations is four.

However, if gravitational field is also subject to some dynamical equations of the form $R_{\mu\nu} = O(1/m^2)$ then the system (55) contains only one non-minimal coupling in the lowest order

$$
\begin{aligned}
&\nabla^2 H_{\mu\nu} - m^2 H_{\mu\nu} + (\xi_3 + 2) R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + O\Big(\frac{1}{m^2}\Big) = 0, \\
&H + O\Big(\frac{1}{m^2}\Big) = 0, \qquad \nabla^\mu H_{\mu\nu} + O\Big(\frac{1}{m^2}\Big) = 0, \\
&R_{\mu\nu} + O\Big(\frac{1}{m^2}\Big) = 0
\end{aligned}
\tag{56}
$$

and is consistent for any its value.

Requirement of causality does not impose any restrictions on the couplings in this order. The characteristic matrix of (55) is non-degenerate, second derivatives enter in the same way as in the flat spacetime, and hence the light cones of the field $H_{\mu\nu}$ described by (55) are the same as in the flat case. Propagation is causal for any values of ξ_1, ξ_2, ξ_3. In higher orders in $1/m^2$ situation becomes more complicated and we expect that requirement of causality may give additional restrictions on the non-minimal couplings.

Concluding this section we would like to stress once more that the theory (51) admits any gravitational background and so no inconsistencies arise if one treats gravity as dynamical field satisfying Einstein equations with the energy - momentum tensor for the field $H_{\mu\nu}$. The action for the system of interacting gravitational field and massive spin 2 field and the Einstein equations for it are:

$$
\begin{aligned}
&S = S_E + S_H, \qquad S_E = -\frac{1}{\kappa^{D-2}} \int d^D x \sqrt{-G} R, \\
&R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = \kappa^{D-2} T^H_{\mu\nu}, \qquad T^H_{\mu\nu} = \frac{1}{\sqrt{-G}} \frac{\delta S_H}{\delta G^{\mu\nu}}
\end{aligned}
\tag{57}
$$

with S_H given by (51). However, making the metric dynamical we change the structure of the second derivatives by means of nonminimal terms $\sim RHH$ which can spoil causal propagation of both metric and massive spin 2 field[5]. This will impose extra restrictions on the parameters of the theory. Also, one can consider additional requirements the theory should fulfill, e.g. tree level unitarity of graviton - massive spin 2 field interaction[22].

4 String theory in background of massive spin 2 field

In this section we will consider sigma-model description of an open string interacting with two background fields – massless graviton $G_{\mu\nu}$ and second rank symmetric tensor field $H_{\mu\nu}$ from the first massive level of the open string spectrum. We will show that effective equations of motion for these fields are of the form (56) and explicitly calculate the coefficient ξ_3 in these equations in the lowest order in α'.

Classical action has the form

$$\begin{aligned} S &= S_0 + S_I = \\ &= \frac{1}{4\pi\alpha'}\int_M d^2z\sqrt{g}g^{ab}\partial_a x^\mu \partial_b x^\nu G_{\mu\nu} + \frac{1}{2\pi\alpha'\mu}\int_{\partial M} edt\ H_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \end{aligned} \tag{58}$$

Here $\mu, \nu = 0, \ldots, D-1$; $a, b = 0, 1$ and we introduced the notation $\dot{x}^\mu = \frac{dx^\mu}{edt}$. The first term S_0 is an integral over two-dimensional string world sheet M with metric g_{ab} and the second S_I represents a one-dimensional integral over its boundary with einbein e. We work in euclidian signature and restrict ourselves to flat world sheets with straight boundaries. It means that both two-dimensional scalar curvature and extrinsic curvature of the world sheet boundary vanish and we can always choose such coordinates that $g_{ab} = \delta_{ab}$, $e = 1$.

Theory has two dimensionful parameters. α' is the fundamental string length squared, D-dimensional coordinates x^μ have dimension $\sqrt{\alpha'}$. Another parameter μ carries dimension of inverse length in two-dimensional field theory (58) and plays the role of renormalization scale. It is introduced in (58) to make the background field $H_{\mu\nu}$ dimensionless. In fact, power of μ is responsible for the number of massive level to which a background field belongs because one expects that open string interacts with a field from n-th massive level through the term

$$\mu^{-n}(\alpha')^{-\frac{n+1}{2}}\int_{\partial M} edt\ \dot{x}^{\mu_1}\ldots\dot{x}^{\mu_{n+1}}H_{\mu_1\ldots\mu_{n+1}}(x)$$

The action (58) is non-renormalizable from the point of view of two-dimensional quantum field theory. Inclusion of interaction with any massive background produces in each loop an infinite number of divergencies and so requires an infinite number of different massive fields in the action. But massive modes from the $n-$th massive level give vertices proportional to μ^{-n} and so they cannot contribute to renormalization of fields from lower levels. Of course, this argument assumes that we treat the theory perturbatively defining propagator for X^μ only by the term with graviton in (58). Now we will use such a scheme to carry out renormalization of (58) dropping all the terms $O(\mu^{-2})$.

Varying (58) one gets classical equations of motion with boundary conditions:

$$
\begin{aligned}
& g^{ab} D_a \partial_b x^\alpha \equiv g^{ab}(\partial_a \partial_b x^\alpha + \Gamma^\alpha_{\mu\nu}(G) \partial_a x^\mu \partial_b x^\nu) = 0, \\
& G_{\mu\nu} \partial_n x^\mu |_{\partial M} - \frac{2}{\mu} \mathcal{D}_t^2 x^\mu H_{\mu\nu} \\
& \qquad + \frac{1}{\mu} \dot{x}^\mu \dot{x}^\lambda (\nabla_\nu H_{\mu\lambda} - \nabla_\mu H_{\nu\lambda} - \nabla_\lambda H_{\mu\nu}) = 0
\end{aligned} \tag{59}
$$

where $\partial_n = n^a \partial_a$, n^a – unit inward normal vector to the world sheet boundary and $\mathcal{D}_t^2 x^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}(G) \dot{x}^\nu \dot{x}^\lambda$.

Divergent part of the one loop effective action has the form

$$
\begin{aligned}
\Gamma^{(1)}_{div} = & -\frac{\mu^{-\varepsilon-1}}{2\pi\varepsilon} \int_{\partial M} dt e(t) \dot{x}^\mu \dot{x}^\nu \left(\nabla^2 H_{\mu\nu} - 2 R_\mu{}^\alpha H_{\alpha\nu} + R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} \right) \\
& + \frac{\mu^{-\varepsilon}}{4\pi\varepsilon} \int_M d^{2+\varepsilon} z \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x^\nu R_{\mu\nu} + O(\mu^{-2})
\end{aligned} \tag{60}
$$

where the terms $O(\mu^{-2})$ give contributions to renormalization of only the second and higher massive levels. Hence one-loop renormalization of the background fields looks like:

$$
\begin{aligned}
\overset{\circ}{G}_{\mu\nu} &= \mu^\varepsilon G_{\mu\nu} - \frac{\alpha' \mu^\varepsilon}{\varepsilon} R_{\mu\nu} \\
\overset{\circ}{H}_{\mu\nu} &= \mu^\varepsilon H_{\mu\nu} + \frac{\alpha' \mu^\varepsilon}{\varepsilon} \left(\nabla^2 H_{\mu\nu} - 2 R^\sigma{}_{(\mu} H_{\nu)\sigma} + R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} \right)
\end{aligned} \tag{61}
$$

with circles denoting bare values of the fields. We would like to stress once more that higher massive levels do not influence the renormalization of any given field from the lower massive levels and so the result (61) represents the full answer for perturbative one-loop renormalization of $G_{\mu\nu}$ and $H_{\mu\nu}$.

Now to impose the condition of Weyl invariance of the theory at the quantum level we calculate the trace of energy momentum tensor in $d = 2 + \varepsilon$ dimension:

$$
T(z) = g_{ab}(z) \frac{\delta S}{\delta g_{ab}(z)} = \frac{\varepsilon \mu^{-\varepsilon}}{8\pi\alpha'} g^{ab}(z) \partial_a x^\mu \partial_b x^\nu G_{\mu\nu} - \frac{\mu^{-1-\varepsilon}}{4\pi\alpha'} H_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \delta_{\partial M}(z) \tag{62}
$$

and perform one-loop renormalization of the composite operators:

$$
\left(\dot{x}^\mu \dot{x}^\nu \overset{\circ}{H}_{\mu\nu} \right)_0 = \mu^{-\varepsilon} \left[\dot{x}^\mu \dot{x}^\nu H_{\mu\nu} \right] \tag{63}
$$

$$
\begin{aligned}
&(g^{ab}\partial_a x^\mu \partial_b x^\nu \overset{\circ}{G}_{\mu\nu})_0 = \mu^\varepsilon \Big[g^{ab}\partial_a x^\mu \partial_b x^\nu (G_{\mu\nu} - \frac{\alpha'}{\varepsilon} R_{\mu\nu}) \Big] \\
&+ \frac{\alpha' \mu^{-1+\varepsilon}}{\varepsilon} \big[H_\alpha{}^\alpha \delta''_{\partial M}(z) + \mathcal{D}_t^2 x^\mu (\nabla_\mu H_\alpha{}^\alpha - 4\nabla^\alpha H_{\alpha\mu}) \delta_{\partial M}(z) \\
&+ \dot{x}^\mu \dot{x}^\nu (\nabla_\mu \nabla_\nu H_\alpha{}^\alpha - 4\nabla^\alpha \nabla_{(\mu} H_{\nu)\alpha} + 2\nabla^2 H_{\mu\nu} - 2R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta}) \delta_{\partial M}(z) \big]
\end{aligned} \tag{64}
$$

Here delta-function of the boundary $\delta_{\partial M}(z)$ is defined as

$$
\int_M \delta_{\partial M}(z) V(z) \sqrt{g(z)} d^2 z = \int_{\partial M} V|_{z \in \partial M} e(t) dt \tag{65}
$$

The renormalized operator of the energy momentum tensor trace is:

$$
\begin{aligned}
8\pi[T] = &- \Big[g^{ab}\partial_a x^\mu \partial_b x^\nu E^{(0)}_{\mu\nu}(x) \Big] + \frac{2}{\mu} \delta_{\partial M}(z) \Big[\dot{x}^\mu \dot{x}^\nu E^{(1)}_{\mu\nu}(x) \Big] \\
&+ \frac{1}{\mu} \delta_{\partial M}(z) \Big[\mathcal{D}_t^2 x^\mu E^{(2)}_\mu(x) \Big] + \frac{1}{\mu} \delta''_{\partial M}(z) \Big[E^{(3)}(x) \Big]
\end{aligned} \tag{66}
$$

where

$$
\begin{aligned}
E^{(0)}_{\mu\nu}(x) &= R_{\mu\nu} + O(\alpha') \\
E^{(1)}_{\mu\nu}(x) &= \nabla^2 H_{\mu\nu} - \nabla^\alpha \nabla_\mu H_{\alpha\nu} - \nabla^\alpha \nabla_\nu H_{\alpha\mu} \\
&\quad - R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + \frac{1}{2} \nabla_\mu \nabla_\nu H_\alpha{}^\alpha - \frac{1}{\alpha'} H_{\mu\nu} + O(\alpha') \\
E^{(2)}_\mu(x) &= \nabla_\mu H_\alpha{}^\alpha - 4\nabla^\alpha H_{\alpha\mu} + O(\alpha') \\
E^{(3)}(x) &= H_\alpha{}^\alpha + O(\alpha')
\end{aligned} \tag{67}
$$

Terms of order $O(\alpha')$ arise from the higher loops contributions.

The requirement of quantum Weyl invariance tells that all $E(x)$ in (67) should vanish and so they are interpreted as effective equations of motion for background fields. They contain vacuum Einstein equation for graviton (in the lowest order in α'), curved spacetime generalization of the mass shell condition for the field $H_{\mu\nu}$ with the mass $m^2 = (\alpha')^{-1}$ and $D+1$ additional constraints on the values of this fields and its first derivatives. Taking into account these constraints and the Einstein equation we can write our final equations arising from the Weyl invariance of string theory in the form:

$$
\begin{aligned}
&\nabla^2 H_{\mu\nu} + R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} - \frac{1}{\alpha'} H_{\mu\nu} + O(\alpha') = 0, \\
&\nabla^\alpha H_{\alpha\nu} + O(\alpha') = 0, \qquad H^\mu{}_\mu + O(\alpha') = 0, \\
&R_{\mu\nu} + O(\alpha') = 0.
\end{aligned} \tag{68}
$$

They coincide with the equations found in the previous section (56) with the value of non-minimal coupling $\xi_3 = -1$.

In fact, Einstein equations should not be vacuum ones but contain dependence on the field $H_{\mu\nu}$ through its energy - momentum tensor $T^H_{\mu\nu}$. Our calculations could not produce this dependence because such dependence is expected to arise only if one takes into account string world sheets with nontrivial topology and renormalizes new divergencies arising from string loops contribution[23].

$$R_{\mu\nu} + O(\alpha') = T^H_{\mu\nu} - \frac{1}{D-2} T^{H\,\alpha}{}_{\alpha}, \tag{69}$$

where explicit form of the lowest contributions to the energy-momentum tensor $T^H_{\mu\nu}$ can be determined only from sigma model on world sheets with topology of annulus.

In order to determine whether the equations (68) can be deduced from an effective lagrangian (and to find this lagrangian) one would need two-loop calculations in the string sigma-model. Two-loop contributions to the Weyl anomaly coefficients $E^{(i)}$ are necessary because the effective equations of motion (67,68) are not the equations directly following from a lagrangian but some combinations of them similar to (55). In order to reverse the procedure of passing from the original lagrangian equations to (55) one would need the next to leading contributions in the conditions for $\nabla^\mu H_{\mu\nu}$ and $H_{\mu\nu}$ (68).

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